university of groningen

Exam Computer Assisted Problem Solving (CAPS)

March 30th 2015 9.00-12.00

This exam is 'closed book'. It is NOT allowed to use a book or lecture notes. Only the use of a simple pocket calculator is allowed. Programmable calculators are not allowed, nor the use of electronic devices (tablet, laptop, etc.).

Always give a clear explanation of your answer. An answer without any computation will not be rewarded, so also copy the computations from your scratch paper.

Write your name and student number on each page!

Free points: 10

1. For the intersection (near $x = 0.5$) of $f_1(x) = e^{x-2}$ and $f_2(x) = x^2$ we have to solve

$$e^{x-2} - x^2 = 0$$

   (a) 5 (1) Give the iteration formula when Newton's method is used for this problem.
   (2) Give 2 pro's and 2 con's of Newton's method for a general problem.

   (b) 5 Determine a $x_{n+1} = g(x_n)$ method with optimal linear convergence factor for this problem, by introduction and optimisation of a parameter α.

   (c) 10 Someone uses the iterative method $x_{n+1} = \sqrt{e^{x_n - 2}}$, with $x_0 = 0.5$. The first 4 iterations are given by

n	x_n
0	0.50000000
1	0.47236655
2	0.46588488
3	0.46437746
4	0.46402759

   (1) Use the convergence theorem to show that this iterative method will converge.
   (2) Determine an error estimate for x_4.
   (3) Calculate an improved solution by means of Steffensen extrapolation.

   (d) 6 Give a complete program (in pseudo code or Matlab-like language) that solves the problem with an accuracy of tol=1E-6, using the iterative method discussed in (c). Use an appropriate error estimate for the stopping criterion.

2. Consider the integral $I = \int_0^2 \frac{1}{\sqrt{2\,x}}\, dx$

   (a) 6 (1) What is the area of the Trapezium at the middle segment if the Trapezoidal method is applied on a grid with 5 sub-intervals (segments).
   (2) Do you expect optimal convergence when the Trapezoidal method is used? Explain why.

P.T.O.

The integral is reformulated as $I = \int_0^1 (8\,x^3)\,dx$.

With the Trapezoidal method the following results are obtained, with $I(n)$ the approximation of I on a grid with n sub-intervals

n	$I(n)$
16	2.00781250
32	2.00195312
64	2.00048828
128	2.00012208

(b) [11] (1) Compute the q-factor and explain that error estimations are allowed. Then give an error estimate for $I(128)$ based on subsequent $I(n)$ values.
(2) Also give the error estimate that follows from the theorem for the global error.
(3) Which estimate is the better one for this integral, theoretically? Explain why.

(c) [6] (1) Compute an improved solution $T_2(128)$ by means of Romberg extrapolation.
(2) Explain how many function evaluations are needed (in total) to be able to compute $T_2(128)$ in a computationally efficient way.

(d) [6] Give a complete program (in pseudo code or Matlab-like language) that solves the problem with an accuracy of tol=1E-6, using the Trapezoidal method. Use an appropriate error estimate for the stopping criterion.

3. Consider for [0 4] the diff. eqn. $y'(x) = -y^{\beta} + (2-\beta)\,x$, with boundary condition $y(0) = 1$. The parameter β will be determined below.

(a) [6] Take $\beta = 1$. Compute the solution at $x = 0.5$:
(1) with the explicit Euler method on a grid with $\Delta x = 0.25$.
(2) with the implicit Euler method on a grid with $\Delta x = 0.5$.

(b) [6] With a 3rd order method the solution for $\beta = 2$ is determined on 2 grids with $\Delta x = 0.5$ and $\Delta x = 0.25$. The result at a selection of x locations is as follows

x_n	$\Delta x = 0.5$	$\Delta x = 0.25$
1.0	0.483144	0.496021
2.0	0.323610	0.330991
3.0	0.243890	0.248521
4.0	0.195838	0.198991

First deduce the appropriate formulas for a 3rd order method, then give for the solution at $x = 3.0$ on the fine grid: (1) an error estimate.
(2) an improved solution by means of extrapolation.

(c) [5] Show that for a general problem, the Trapezoidal method for differential equations can be deduced by means of combination of the implicit and explicit Euler method.

(d) [8] Give a complete program (in pseudo code or Matlab-like language) that solves the problem for $\beta = 3$ with an accuracy of tol=1E-6, using the explicit Euler method. Use an appropriate error estimate for the stopping criterion.

4. Consider for [0 4] the diff. eqn. $y''(x) + e^x y(x) = x^2$, with boundary conditions $y(0) = 1$ and $y(4) = 2$.

(a) [7] Describe the matrix and rhs-vector when the problem is solved by means of the matrix method, using the standard [1 -2 1]-formula for $y''(x)$.

(b) [3] Which modification do you have to make to the system when the boundary condition at $x = 4$ is changed into $y'(4) = 1$?

Total: [100]